Some definite logarithmic integrals from Euler sums, and other integration results

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Abstract

We present explicit expressions for multi-fold logarithmic integrals that are equivalent to sums over polygamma functions at integer argument. Such relations find application in perturbative quantum field theory, quantum chemistry, analytic number theory, and elsewhere. The analysis includes the use of properties of a variety of special functions.

Key words and phrases

dilogarithm function, digamma function, polygamma function, harmonic numbers, Euler sums, polylogarithm function, Hurwitz zeta function, Riemann zeta function, generalized harmonic numbers, generalized hypergeometric function

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Introduction

It is well known that sums over polygamma functions at integer arguments, as well as a vast array of logarithmic and polylogarithmic integrals, are useful for applications in quantum field theory and elsewhere [4, 5, 7]. Additionally motivated by multiple-logarithm integrals occurring very recently in quantum chemistry [9], (see Propositions 2 and 6) we provide details for certain one-dimensional sums over polygamma functions. Our treatment overlaps several areas of analysis, special function theory, analytic number theory [3, 12], and integration [6]. This is evidenced in the first Proposition for $(k+1)$-logarithmic integrals that evaluate as Euler sums and hence explicitly in terms of Riemann zeta function values at the positive integers. Moreover, polygamma functions at the positive integers are equivalent to the Hurwitz zeta function $\zeta(s, a)$ at the positive integers for both arguments.

Our work can be extended to relations with the Lerch zeta function $\Phi(z, s, a)$. However, we restrict attention here to relations of much more immediate applications interest.

We let ζ be the Riemann zeta function, $\text{Li}_s(z)$ the polylogarithm function [11], and $H_n \equiv \sum_{k=1}^n 1/k$ the nth harmonic number. Material on special functions may be found in [1, 2, 8, 5], and especially on the dilogarithm Li_2 and polylogarithm functions in Lewin [10, 11].

Our first result evaluates $(k+1)$-fold logarithmic integrals in terms of sums of ζ values of weight $k+2$. These integrals are equivalent to a non alternating sum of polygamma function values. We then evaluate $(j+1)$-fold logarithms integrals that

correspond to an alternating sum of polygamma function values, Proposition 2 giving the details of the $j = 2$ case. As a component of Proposition 4 for the integral of a polylogarithm function and a rational factor, we explicitly obtain alternating parameterized polygamma function summations in two different ways. Proposition 5 subsumes Proposition 1 in that all integer powers of the logarithmic factors are allowed. In Proposition 6, we obtain parameterized polygamma sums and their equivalent multi-fold logarithmic integrals with rational factor.

Definite logarithmic integrals from Euler sums

The definite integrals (44) of Coffey [4] are special cases of the following.

Proposition 1. For integers $k \geq 1$ we have

$$\int_0^1 \frac{\ln(1-t)}{1-t} \ln^k t \, dt = \frac{(-1)^{k+1}}{2} k! \left[(k+1)\zeta(k+2) - \sum_{n=1}^{k-1} \zeta(k-n+1)\zeta(n+1) \right]. \quad (1)$$

Proof. Integrating by parts, we have

$$\int_0^1 \frac{\ln(1-t)}{1-t} \ln^k t \, dt = \frac{k}{2} \int_0^1 \frac{\ln^{k-1} t}{t} \ln^2(1-t) \, dt$$

$$= (-1)^{k+1} k! \sum_{r=1}^\infty \frac{H_r}{(r+1)^{k+1}}. \quad (2)$$

Here we have used the standard generating function

$$\frac{1}{2} \ln^2(1-x) = \sum_{k=1}^\infty \frac{H_k}{k+1} x^{k+1}, \quad |x| < 1. \quad (3)$$

This equation immediately follows from integrating the standard generating function for $\text{Li}_1(z) = -\ln(1-z)$. In obtaining (2) we have also interchanged summation and integration, justified on the basis of absolute convergence, and used the Γ function

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integral

$$\int_0^1 t^r \ln^{k-1} t \ dt = (-1)^{k-1} \int_0^\infty u^{k-1} e^{-(r+1)u} du = (-1)^{k-1} \frac{(k-1)!}{(r+1)^k}. \tag{4}$$

We now use the functional equation of harmonic numbers, $H_r = H_{r-1} + 1/r$, to determine

$$\sum_{r=1}^\infty \frac{H_r}{(r+1)^{k+1}} = \sum_{r=1}^\infty \frac{H_r}{r^{k+1}} - \zeta(k+2). \tag{5}$$

We then apply Euler's relation for integers $m \geq 2$

$$2 \sum_{n=1}^\infty \frac{H_n}{n^m} = (m+2)\zeta(m+1) - \sum_{n=1}^{m-2} \zeta(m-n)\zeta(n+1), \tag{6}$$

at $m = k+1$, and the Proposition follows.

Further example. We have

$$\int_0^1 \frac{\ln(1-t)}{1-t} \ln^5 t \ dt = -\frac{4}{3}\pi^4 \zeta(3) - 20\pi^2 \zeta(5) + 360\zeta(7). \tag{7}$$

Remarks. There are several other representations for the Euler sums on the right side of (2). By using the inverse Laplace transform of $(r+1)^{-(k+1)}$ [cf. (4)] and interchanging summation and integration we have for $k \geq 1$

$$(-1)^{k+1} k! \sum_{r=1}^\infty \frac{H_r}{(r+1)^{k+1}} = (-1)^k \int_0^\infty s^k \frac{\ln(1-e^{-s})}{e^s - 1} ds. \tag{8}$$

In the Proposition, we have an example of integrating an integrand of logarithmic weight $k+1$ that evaluates in terms of zeta values of weight $k+2$.

Relations with regard to sums over polygamma functions $\psi^{(j)}$ we largely omit. However, we may readily verify that for $k \geq 1$,

$$(-1)^{k+1} k! \sum_{r=1}^\infty \frac{H_r}{r^{k+1}} = \sum_{\ell=1}^\infty \frac{\psi^{(k)}(\ell)}{\ell}. \tag{9}$$

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Upon interchanging sums, we have

$$\sum_{r=1}^{\infty} \frac{H_r}{r^{k+1}} = \sum_{r=1}^{\infty} \frac{1}{r^{k+1}} \sum_{\ell=1}^{r} \frac{1}{\ell}$$

$$= \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sum_{r=\ell}^{\infty} \frac{1}{r^{k+1}} = \sum_{\ell=1}^{\infty} \frac{1}{\ell} \sum_{r=0}^{\infty} \frac{1}{(r+\ell)^{k+1}}$$

$$= \sum_{\ell=1}^{\infty} \frac{1}{\ell} \zeta(k+1, \ell) = \frac{(-1)^{k+1}}{k!} \sum_{\ell=1}^{\infty} \frac{\psi^{(k)}(\ell)}{\ell}. \tag{10}$$

Other integral expressions

We first give expressions for the integral

$$I_2 \equiv \int_0^1 \ln(1+t) \frac{\ln^2 t}{1-t} dt \simeq 0.345690556478. \tag{11}$$

We have

$$I_2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \psi''(n+1)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \psi''(n) - \frac{7}{360} \pi^4. \tag{12}$$

In obtaining the second line here we used the functional equation of the tetragamma function, $\psi''(x+1) = \psi''(x) + 2/x^3$, along with the alternating form of the Riemann zeta function.

Furthermore, we have

Proposition 2

$$I_2 = \sum_{k=1}^{\infty} (-1)^k \int_0^{\infty} e^{-kt} \psi''(k) dt - \frac{7}{360} \pi^4$$

$$= \frac{7}{2} \ln 2 \, \zeta(3) - \frac{19}{720} \pi^4. \tag{13}$$

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Proof. Here, with Li$_3$ the trilogarithm function, we have used the integral

$$\int_0^\infty \frac{\text{Li}_3(-e^{-t})}{1+e^t} dt = \int_0^1 \frac{\text{Li}_3(-u)}{1+u} du = \frac{\pi^4}{288} - \frac{3}{4} \ln 2\, \zeta(3). \tag{14}$$

We also recall the following relations. We have

$$\psi''(x) = -2 \sum_{n=0}^\infty \frac{1}{(n+x)^3}. \tag{15}$$

Then we have

$$\sum_{k=1}^\infty (-1)^k e^{-kt} \psi''(k) = -2 \sum_{k=1}^\infty (-1)^k e^{-kt} \sum_{n=0}^\infty \frac{1}{(n+k)^3}$$

$$= -2 \sum_{k=1}^\infty (-1)^k e^{-kt} \sum_{n=k}^\infty \frac{1}{n^3} = -2 \sum_{n=1}^\infty \frac{1}{n^3} \sum_{k=1}^n (-1)^k e^{-kt}$$

$$= \frac{-2}{1+e^t} \sum_{n=1}^\infty \frac{1}{n^3} [(-1)^n e^{-nt} - 1]$$

$$= \frac{-2}{1+e^t} [\text{Li}_3(-e^{-t}) - \zeta(3)]. \tag{16}$$

For the integral (14) we employ integration by parts twice:

$$\int_0^1 \text{Li}_3(-u) \left(\frac{d}{du} \ln(1+u) \right) du = -\int_0^1 \frac{\text{Li}_2(-u)}{u} \ln(1+u) du - \frac{3}{4} \ln 2\, \zeta(3)$$

$$= -\int_0^1 \text{Li}_2(-u) \left(\frac{d}{du} \text{Li}_2(-u) \right) du - \frac{3}{4} \ln 2\, \zeta(3)$$

$$= -\frac{1}{2} \text{Li}_2^2(-u)|_0^1 - \frac{3}{4} \ln 2\, \zeta(3)$$

$$= -\frac{1}{2} \left(-\frac{\pi^2}{12} \right)^2 - \frac{3}{4} \ln 2\, \zeta(3). \tag{17}$$

Remarks. Alternatively, we may employ the generating function

$$\sum_{n=1}^\infty (-1)^n H_n z^n = -\frac{\ln(1+z)}{1+z}. \tag{18}$$

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Then we obtain

$$I_2 = -2\sum_{n=1}^{\infty}(-1)^n H_n \left[\frac{1}{(n+1)^3} + \psi''(n+1)\right]$$

$$= -2\sum_{n=1}^{\infty}(-1)^n H_n \left[\frac{1}{(n+1)^3} + \frac{2}{n^3} + \psi''(n)\right]. \tag{19}$$

The integral I_2 appears in a semi-classical treatment of neutral atoms and diatomic molecules [9]. This theory employs improved Thomas-Fermi models and provides chemically accurate results. It enables a better understanding of the variation of properties within, for instance, a series of homologous compounds by including the number of electrons as a variable. It is within this kind of framework that various sums and integrals have arisen and require closed forms for the theory's most exact expression.

We have the following Corollary from Proposition 2.

Corollary 1. A closed form for the integral $\int_0^1 \ln(1-t)\frac{\ln^2 t}{1+t}dt$ determines one for the integral $\int_0^1 \frac{\ln t}{t}\ln(1-t)\ln(1+t)dt$, and vice versa.

Proof. By integrating by parts, we have from the definition (11)

$$I_2 = \int_0^1 \ln(1-t)\left[\frac{\ln^2 t}{1+t} + 2\frac{\ln t}{t}\ln(1+t)\right]dt.$$

Since I_2 is known, the result follows.

Expressions for the integrals of Corollary 1 follow from the considerations given in the Appendix.

<center>**Generalization**</center>

It is evident that the method of Proposition 2 extends to a family of integrals for the "lower sign" of (43) of [4]. We have

$$\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}}. \tag{20}$$

Then we obtain

$$\sum_{k=1}^{\infty} (-1)^k e^{-kt} \psi^{(n)}(k) = \frac{(-1)^{n+1} n!}{1+e^t} \left[\mathrm{Li}_{n+1}(-e^{-t}) - \zeta(n+1) \right]. \tag{21}$$

We use this in the proof of

Proposition 3. Put

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \psi^{(j)}(n+1) = \int_0^1 \ln^j t \frac{\ln(1+t)}{1-t} dt \equiv I_j. \tag{22}$$

Then for integers $j \geq 1$ we have

$$I_j = (-1)^{j+1} j! \int_0^1 \frac{\mathrm{Li}_{j+1}(-u)}{1+u} du + (-1)^j j! [\ln 2\, \zeta(j+1) + (2^{-j-1} - 1)\zeta(j+2)]. \tag{23}$$

Proof. The equality of the summation and integration in (22) may be verified in many ways, including simply expanding the factor $\ln(1+t)$ of the integrand in power series:

$$I_j = \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} \int_0^1 t^{\ell} \frac{\ln^j t}{1-t} dt$$

$$= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell+1}}{\ell} (-1)^j j! \zeta(j+1, \ell+1)$$

$$= \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{\ell} \psi^{(j)}(\ell+1). \tag{24}$$

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The interchange of summation and integration is justified by absolute convergence.

We have the functional equation

$$\psi^{(j)}(x+1) = \psi^{(j)}(x) + \frac{(-1)^j j!}{x^{j+1}}. \tag{25}$$

Using this relation and proceeding as in the proof of the previous Proposition, we have

$$I_j = \sum_{n=1}^{\infty}(-1)^n \int_0^{\infty} e^{-nt}\psi^{(j)}(n)dt + (-1)^j j! \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{j+2}}$$

$$= \sum_{n=1}^{\infty}(-1)^n \int_0^{\infty} e^{-nt}\psi^{(j)}(n)dt + (-1)^j j!(2^{-j-1} - 1)\zeta(j+2), \tag{26}$$

where we used the alternating form of the Riemann zeta function, $\mathrm{Li}_j(-1) = (2^{1-j} - 1)\zeta(j)$. From relations (21) and (26) we obtain

$$(-1)^{j+1}j! \int_0^{\infty} \frac{1}{1+e^t}\left[\mathrm{Li}_{j+1}(-e^{-t}) - \zeta(j+1)\right]dt + (-1)^j j!(2^{-j-1} - 1)\zeta(j+2)$$

$$= (-1)^{j+1}j! \int_0^1 \frac{[\mathrm{Li}_{j+1}(-u) - \zeta(j+1)]}{1+u}du + (-1)^j j!(2^{-j-1} - 1)\zeta(j+2), \tag{27}$$

from which the Proposition follows.

Remarks. Not surprisingly, the polylogarithmic integrals of Proposition 3 may be written in terms of sums over generalized harmonic numbers $H_n^{(r)} \equiv \sum_{k=1}^{n} 1/k^r$. We have the generating function

$$\sum_{n=1}^{\infty}(-1)^n H_n^{(r)} z^n = \frac{\mathrm{Li}_r(-z)}{1+z}, \quad |z| < 1. \tag{28}$$

Then using the functional equation $H_n^{(j+1)} = H_{n+1}^{(j+1)} - 1/(n+1)^{j+1}$ we obtain

$$\int_0^1 \frac{\mathrm{Li}_{j+1}(-u)}{1+u}du = -\sum_{n=2}^{\infty} \frac{(-1)^n}{n}H_n^{(j+1)} + 1 + (2^{-j-1} - 1)\zeta(j+2). \tag{29}$$

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We next present an evaluation of the polylogarithmic integrals of Proposition 3. We have

Proposition 4. For integers $j \geq 1$ we have (a)

$$\int_0^1 \frac{\mathrm{Li}_{j+1}(-u)}{1+u} du = (-1)^j \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{j+1}} \Phi(-1,1,k+1) + [(-1)^j + 1](2^{-j} - 1)\ln 2\, \zeta(j+1)$$

$$+ (-1)^j \sum_{m=1}^{j-1} (-1)^m (2^{-m} - 1)\zeta(m+1)(1 - 2^{m-j})\zeta(j+1-m). \tag{30}$$

(b) Of note, when j is odd, we have

$$\int_0^1 \frac{\mathrm{Li}_{j+1}(-u)}{1+u} du = (-1)^j \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{j+1}} \Phi(-1,1,k+1). \tag{31a}$$

When j is even, we have

$$\int_0^1 \frac{\mathrm{Li}_{j+1}(-u)}{1+u} du = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{j+1}} \Phi(-1,1,k+1) + 2(2^{-j} - 1)\ln 2\, \zeta(j+1)$$

$$+2\sum_{m=1}^{j/2-1} (-1)^m (2^{-m}-1)\zeta(m+1)(1-2^{m-j})\zeta(j+1-m)+(-1)^{j/2-1}(1-2^{-j/2})^2\zeta^2(j/2+1).$$

$$\tag{31b}$$

(c) We have the alternative representation

$$\int_0^1 \frac{\mathrm{Li}_{j+1}(-u)}{1+u} du = \ln 2\, \zeta(j+1) - \frac{1}{j!} \int_0^{\infty} \frac{t^j e^t}{e^t - 1} \ln(1 + e^{-t}) dt \tag{32}$$

$$= \ln 2\, \zeta(j+1) + [2^{-(j+1)} - 1]\zeta(j+2) + \frac{(-1)^{j-1}}{j!} \int_0^1 \ln^j v \frac{\ln(1+v)}{1-v} dv. \tag{33}$$

Proof. By expanding the integrand factor $(1+u)^{-1}$ as a geometric series, by integrating term-by-term we have

$$\int_0^1 \frac{\mathrm{Li}_{j+1}(-u)}{1+u} du = \sum_{k=0}^{\infty} (-1)^k \int_0^1 u^k \mathrm{Li}_{j+1}(-u) du$$

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$$= \sum_{k=0}^{\infty} (-1)^k \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{j+1}} \int_0^1 u^{k+n} du$$

$$= \sum_{k=0}^{\infty} (-1)^k \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{j+1}} \frac{1}{(k+n+1)}$$

$$= \sum_{k=0}^{\infty} (-1)^k S_j(k), \tag{34}$$

where we have defined for $z \in C/\{-2, -3, \ldots\}$,

$$S_j(z) \equiv \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{j+1}} \frac{1}{(n+z+1)}. \tag{35}$$

We next explicitly determine $S_j(z)$ by writing a recursion relation for it. We have

$$S_j(z) = \frac{1}{(z+1)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^j} \left(\frac{1}{n} - \frac{1}{n+z+1} \right), \tag{36}$$

and find

$$S_j(z) = -\frac{S_{j-1}(z)}{z+1} + \frac{1}{(z+1)} (2^{-j} - 1)\zeta(j+1), \tag{37a}$$

with

$$S_0(z) = \frac{1}{(z+1)} [\Phi(-1, 1, z+2) - \ln 2]. \tag{37b}$$

Iteration of the recurrence (37a) gives

$$S_j(z) = (-1)^j \frac{S_0(z)}{(z+1)^j} + \frac{(-1)^j}{(z+1)^{j+1}} \sum_{m=1}^{j} (-1)^m (2^{-m} - 1)(z+1)^m \zeta(m+1). \tag{38}$$

We next substitute (38) in (34) and perform sums with the alternating form of the Riemann zeta function,

$$\sum_{k=0}^{\infty} (-1)^k S_j(k) = (-1)^j \left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{j+1}} \Phi(-1, 1, k+1) - \ln 2(1 - 2^{-j})\zeta(j+1) \right]$$

$$+ (-1)^j \sum_{m=1}^{j} (-1)^m (2^{-m} - 1)\zeta(m+1)(1 - 2^{m-j})\zeta(j+1-m). \tag{39}$$

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We apply the property

$$\lim_{x \to 0}(1 - 2^x)\zeta(1 - x) = \ln 2, \tag{40}$$

for the $m = j$ term in the second line of (39). Then combining $(2^{-j} - 1)\ln 2\,\zeta(j+1)$ terms gives the the final form of part (a) of the Proposition.

For part (c) we employ the integral representation

$$\mathrm{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - z} dt, \tag{41}$$

and interchange integrations. By using partial fractions we have

$$-\int_0^1 \frac{u\,du}{(u+1)(u+e^t)} = \frac{1}{1 - e^t} \int_0^1 \left(\frac{e^t}{u + e^t} - \frac{1}{u+1} \right) du = \frac{1}{e^t - 1}[\ln 2 + te^t - e^t \ln(1+e^t)]. \tag{42}$$

Then using (41) for the special case of $z = 1$ gives (32). With the change of variable $t = -\ln v$, the use of partial fractions,

$$\int_0^1 \frac{\mathrm{Li}_{j+1}(-u)}{1 + u} du = \ln 2\,\zeta(j+1) + \frac{(-1)^j}{j!} \int_0^1 \left(\frac{1}{v} + \frac{1}{1 - v} \right) \ln^j v \ln(1 + v) dv, \tag{43}$$

and lastly integration by parts we obtain (33).

Remarks. Given the definition (22) of I_j, it is seen that (33) is equivalent to (23).

We may note the limit relation

$$\lim_{s \to \infty} \int_0^1 \frac{\mathrm{Li}_{s+1}(-u)}{1 + u} du = -\int_0^1 \frac{u}{1 + u} du = \ln 2 - 1. \tag{44}$$

The value $\Phi(-1, 1, 2) = 1 - \ln 2$ and (35)-(36) reduce properly to the Proposition [4] (p. 96-97; (59), (61)) at $z = 0$. We have

$$\lim_{j \to \infty} S_j(z) = -\frac{1}{z + 2}. \tag{45}$$

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Another method to determine the sums $S_j(z)$ proceeds from the relation

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+b)} \frac{1}{(n+z)} = \frac{1}{(b-z)}[\Phi(-1,1,b+1) - \Phi(-1,1,z+1)]. \qquad (46)$$

This equation follows simply from partial fractions and the series definition of the Lerch zeta function. The terms on the right side of (41) are nothing but a form of the alternating Hurwitz zeta function, as

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+b} = -\Phi(-1,1,b+1) = \frac{1}{2}\left[\psi\left(\frac{b+1}{2}\right) - \psi\left(\frac{b}{2}+1\right)\right]. \qquad (47)$$

We have

$$\frac{\partial}{\partial b}\Phi(-1,1,b+1) = \frac{1}{4}\left[\zeta\left(2,\frac{b}{2}+1\right) - \zeta\left(2,\frac{b+1}{2}\right)\right], \qquad (48)$$

and $\partial_a\zeta(s,a) = -s\zeta(s+1,a)$, giving

$$\frac{\partial^j}{\partial b^j}\Phi(-1,1,b+1) = (-1)^{j+1}\frac{j!}{2^{j+1}}\left[\zeta\left(j+1,\frac{b}{2}+1\right) - \zeta\left(j+1,\frac{b+1}{2}\right)\right]. \qquad (49)$$

We then have

$$\frac{\partial^j}{\partial b^j}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+b)}\frac{1}{(n+z)} = (-1)^j j!\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+b)^{j+1}}\frac{1}{(n+z)}$$

$$= \frac{\partial^j}{\partial b^j}\frac{1}{(b-z)}[\Phi(-1,1,b+1) - \Phi(-1,1,z+1)]$$

$$= \sum_{m=0}^{j}\binom{j}{m}\left[\left(\frac{\partial}{\partial b}\right)^{j-m}\frac{1}{(b-z)}\right]\left(\frac{\partial}{\partial b}\right)^m\Phi(-1,1,b+1) - (-1)^j j!\frac{\Phi(-1,1,b+1)}{(b-z)^{j+1}}$$

$$= \sum_{m=0}^{j}\binom{j}{m}\frac{(j-m)!(-1)^{j-m}}{(b-z)^{j-m+1}}\frac{(-1)^{m+1}m!}{2^{m+1}}\left[\zeta\left(m+1,\frac{b}{2}+1\right) - \zeta\left(m+1,\frac{b+1}{2}\right)\right]$$

$$-(-1)^j j!\frac{\Phi(-1,1,b+1)}{(b-z)^{j+1}}$$

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$$= j! \sum_{m=0}^{j} \frac{(-1)^{j-m}}{(b-z)^{j-m+1}} \frac{(-1)^{m+1}}{2^{m+1}} \left[\zeta \left(m+1, \frac{b}{2}+1 \right) - \zeta \left(m+1, \frac{b+1}{2} \right) \right]$$

$$-(-1)^j j! \frac{\Phi(-1,1,b+1)}{(b-z)^{j+1}}. \tag{50}$$

We now put $b = 0$ and cancel factors $(-1)^j j!$ on both sides of (50), so that

$$S_j(z-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{j+1}} \frac{1}{(n+z)} = \sum_{m=0}^{j} \frac{(-1)^{j-m}}{(-z)^{j-m+1}} \frac{(-1)^{m+1}}{2^{m+1}} \left[\zeta \left(m+1, \frac{b}{2}+1 \right) - \zeta \left(m+1, \frac{b+1}{2} \right) \right]$$

$$- \frac{\Phi(-1,1,b+1)}{(-z)^{j+1}}. \tag{51}$$

By using the relation $\zeta(s, 1/2) = (2^s - 1)\zeta(s)$, we have

$$S_j(z-1) = \frac{(-1)^j}{z^{j+1}} \sum_{m=0}^{j} (-1)^m z^m (2^{-m} - 1)\zeta(m+1) + \frac{(-1)^j}{z^{j+1}} \Phi(-1,1,z+1)$$

$$= \frac{(-1)^j}{z^{j+1}} \sum_{m=1}^{j} (-1)^m z^m (2^{-m} - 1)\zeta(m+1) + \frac{(-1)^{j+1}}{z^{j+1}} \ln 2 + \frac{(-1)^j}{z^{j+1}} \Phi(-1,1,z+1), \tag{52}$$

where we again applied the property (40) for the $m = 0$ term. We recover the expression (38) for $S_j(z)$.

We may generalize the sum of (35) to

$$S_j(t, z) \equiv \sum_{n=1}^{\infty} \frac{t^n}{n^{j+1}} \frac{1}{(n+z+1)}, \qquad |t| \le 1, \tag{53}$$

with

$$S_0(t, z) = \frac{1}{z+1}[-t\Phi(t, 1, z+2) - \ln(1-t)]. \tag{54}$$

We may explicitly determine $S_j(t, z)$. We do so, and record this as the following Lemma, as this result in the special case of $t = -1$ could have been used in the proof

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of Proposition 4. We have

Lemma 1. For integers $j \geq 1$ we have

$$S_j(t, z) = \frac{(-1)^j}{(z+1)^j} S_0(t, z) + \frac{(-1)^j}{(z+1)^{j+1}} \sum_{m=1}^{j} (-1)^m (z+1)^m \mathrm{Li}_{m+1}(t). \qquad (55)$$

Proof. We have

$$S_j(t, z) = \frac{1}{(z+1)} \sum_{n=1}^{\infty} \frac{t^n}{n^j} \left(\frac{1}{n} - \frac{1}{n+z+1} \right), \qquad (56)$$

and find the recurrence

$$S_j(t, z) = -\frac{S_{j-1}(t, z)}{z+1} + \frac{1}{z+1} \mathrm{Li}_{j+1}(t). \qquad (57)$$

Iterating this relation yields the Lemma.

Furthermore, we have the following extension subsuming Lemma 1. We have

Lemma 2. Put

$$S_j(t, z, a) \equiv \sum_{n=1}^{\infty} \frac{t^n}{(n+a)^{j+1}} \frac{1}{(n+z+1)}, \qquad |t| \leq 1, \qquad (58)$$

with

$$S_0(t, z, a) = \frac{t}{z-a+1} [\Phi(t, 1, a+1) - \Phi(t, 1, z+2)]. \qquad (59)$$

Then we have for integers $j \geq 1$

$$S_j(t, z, a) = \frac{(-1)^j}{(z-a+1)^j} S_0(t, z, a) + \frac{(-1)^j t}{(z-a+1)^{j+1}} \sum_{m=1}^{j} (-1)^m (z-a+1)^m \Phi(t, m+1, a+1).$$
$$(60)$$

Proof. The proof is very similar to that for the preceding Lemma and is omitted.

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We next show that a generalization of Proposition 1 to all integer powers of the logarithmic factors can be evaluated in terms of zeta values at the integers. We have

Proposition 5. Put the function

$$z(x,y) \equiv \sum_{k,j=1}^{\infty} (-1)^{k+j+1} \frac{(k+j-1)!}{k!j!} \zeta(k+j)x^k y^j. \tag{61}$$

Then we have

$$\int_0^1 \frac{\ln^k t}{t} \ln^j(1-t) \ dt = \left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^j \Bigg|_{x=y=0} \frac{1}{x} \exp(z). \tag{62}$$

Proof. We make use of the Beta function integral for Re $x > 0$ and Re $y > -1$,

$$\int_0^1 t^{x-1}(1-t)^y dt = B(x, y+1) = \frac{\Gamma(x)\Gamma(y+1)}{\Gamma(x+y+1)}$$

$$= \frac{\Gamma(x+1)\Gamma(y+1)}{x\Gamma(x+y+1)}. \tag{63}$$

Then we have

$$\int_0^1 \frac{\ln^k t}{t} \ln^j(1-t) \ dt = \left(\frac{\partial}{\partial x}\right)^k \left(\frac{\partial}{\partial y}\right)^j \Bigg|_{x=y=0} \frac{1}{x} e^z, \tag{64}$$

where we put $z(x,y) = \ln\Gamma(x+1) + \ln\Gamma(y+1) - \ln\Gamma(x+y+1)$. For $k \geq 1$ and $j \geq 1$ we have

$$\partial_x^k \partial_y^j z = -\psi^{(k+j-1)}(x+y+1)$$

$$= (-1)^{k+j+1}(k+j-1)!\zeta(k+j, x+y+1), \tag{65}$$

where we used relation (20). Noting that $z(0,0) = z(x,0) = z(0,y) = 0$, we have the Maclaurin series

$$z(x,y) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \frac{\partial_x^k \partial_y^j|_{x=y=0} z}{k!j!} x^k y^j = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} (-1)^{k+j+1} \frac{(k+j-1)!}{k!j!} \zeta(k+j)x^k y^j. \tag{66}$$

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The Proposition follows.

Remark. In the Proposition we have developed a 2-variable generating function. A generalization for the integrals

$$\int_0^1 \frac{\ln^j t}{t} \ln^k(1-t) \ln^\ell(1-\alpha t) \; dt \equiv I_{jk\ell}$$

is for Re $u > 0$, Re $v > -1$,

$$\sum_{j,k,\ell=0}^{\infty} \frac{u^j v^k(-w)^\ell}{j!k!\ell!} \int_0^1 \frac{\ln^j t}{t} \ln^k(1-t) \ln^\ell(1-\alpha t) \; dt = \int_0^1 t^{u-1}(1-t)^v(1-\alpha t)^{-w} dt$$

$$= B(u, v+1) \; {}_2F_1(u, w; 1+u+v; \alpha), \tag{67}$$

where ${}_2F_1$ is the Gauss hypergeometric function. I.e., we have

$$I_{jk\ell} = \left(\frac{\partial}{\partial u}\right)^j \left(\frac{\partial}{\partial v}\right)^k \left(-\frac{\partial}{\partial w}\right)^\ell \Bigg|_{u=v=w=0} B(u, v+1) \; {}_2F_1(u, w; 1+u+v; \alpha). \tag{68}$$

Another generalization of Proposition 1 is to consider logarithmic integrals with sign alteration of the integrand. For this we let ${}_kF_\ell$ be the generalized hypergeometric function (e.g., [2]). We have

Proposition 6. Let $|z| \leq 1$ and $p \geq 1$ be an integer. Then we have (a)

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \psi(k) z^k = -\gamma \mathrm{Li}_p(-z) + z \frac{\partial}{\partial \beta}\Bigg|_{\beta=1} {}_{p+2}F_{p+1}(1, \ldots, 1; 2, \ldots, 2, \beta; -z), \tag{69}$$

and (b)

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \psi(k) = -\gamma(1 - 2^{1-p})\zeta(p) + \frac{\partial}{\partial \beta}\Bigg|_{\beta=1} {}_{p+2}F_{p+1}(1, \ldots, 1; 2, \ldots, 2, \beta; -1). \tag{70}$$

Proof. Our proof provides an alternative integral representation. We have

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p} \psi(k) z^k = \frac{1}{\Gamma(p)} \sum_{k=1}^{\infty} (-1)^k \psi(k) z^k \int_0^\infty t^{p-1} e^{-kt} dt$$

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$$= \frac{z}{\Gamma(p)} \int_0^\infty t^{p-1} \frac{[\gamma + \ln(1 + ze^{-t})]}{z + e^t} dt$$

$$= \frac{z}{\Gamma(p)} \int_0^1 \ln^{p-1}\left(\frac{1}{v}\right) \frac{[\gamma + \ln(1 + zv)]}{1 + zv} dv$$

$$= -\gamma \mathrm{Li}_p(-z) - \frac{z}{\Gamma(p)} \sum_{n=1}^\infty (-1)^n H_n z^n \int_0^1 v^n \ln^{p-1}\left(\frac{1}{v}\right) dv, \tag{71}$$

where we have used (18). Carrying out the integral gives

$$\sum_{k=1}^\infty \frac{(-1)^k}{k^p} \psi(k) z^k = -\gamma \mathrm{Li}_p(-z) - z \sum_{n=1}^\infty \frac{(-1)^n H_n z^n}{(n+1)^p}. \tag{72}$$

We now consider the function

$$_{p+2}F_{p+1}(1,\ldots,1;2,\ldots,2,\beta;-z) = \sum_{j=0}^\infty \frac{(1)_j^{p+2}}{(2)_j^p} \frac{1}{(\beta)_j} \frac{(-z)^j}{j!} = \sum_{j=0}^\infty \frac{1}{(j+1)^p} \frac{j!}{(\beta)_j} (-z)^j, \tag{73}$$

where the Pochhammer symbol $(w)_n = \Gamma(w+n)/\Gamma(w)$, giving

$$\frac{\partial}{\partial \beta} {}_{p+2}F_{p+1}(1,\ldots,1;2,\ldots,2,\beta;-z) = \sum_{j=1}^\infty \frac{1}{(j+1)^p} \frac{j!}{(\beta)_j} (-z)^j [\psi(\beta) - \psi(\beta+j)]. \tag{74}$$

Therefore, we find

$$\frac{\partial}{\partial \beta}\bigg|_{\beta=1} {}_{p+2}F_{p+1}(1,\ldots,1;2,\ldots,2,\beta;-z) = -\sum_{j=1}^\infty \frac{(-z)^j}{(j+1)^p}[\gamma + \psi(j+1)] = -\sum_{j=1}^\infty \frac{H_j}{(j+1)^p}(-z)^j. \tag{75}$$

Combining (72) and (75) yields part (a) of the Proposition. Part (b) follows at $z = 1$ by using the relation $\mathrm{Li}_p(-1) = -(1 - 2^{1-p})\zeta(p)$.

Remarks. When $z = 1$ and $p = 1$ in Proposition 5, the term $\gamma(1 - 2^{1-p})\zeta(p)$ gives $-\gamma \ln 2$.

The special case of $z = 1$ and $p = 5$ occurs in quantum chemistry calculations for multi-electron atoms [9].

Equation (72) may be directly obtained from the defining sum on the left side of (69), but this route omits the integral representation that may be of interest in its own right.

Higher order derivatives of $_kF_\ell$ may be used to evaluate sums with generalized harmonic numbers. However, this leads to nonlinear sums. As an example, two derivatives of the generalized hypergeometric function of Proposition 6 generates not just $H_j^{(2)}$ but H_j^2.

An extension of Proposition 6 is to consider sums of the form for Re $a \geq 0$

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{(k+a)^p} \psi(k) z^k = \frac{\gamma z}{\Gamma(p)} \int_0^1 v^a \frac{\ln^{p-1}(1/v)}{1+zv} dv - z \sum_{n=1}^{\infty} \frac{(-1)^n H_n z^n}{(n+a+1)^p}. \tag{76}$$

Appendix: Other triple-logarithm integrals

Here we consider the integrals of Corollary 1. We let $\gamma = -\psi(1)$ be the Euler constant, and recall that the integral I_2 is defined in (11). We first have

Proposition A1. We have

$$\int_0^1 \ln(1-x) \frac{\ln^2 x}{1+x} dx = -\frac{7\pi^4}{180} - \frac{3}{2}\gamma\zeta(3) + 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j^3}[\psi(j) - j\psi'(j)] + I_2. \tag{A.1}$$

Proof. We first note that the integral

$$\int_0^1 x^a \ln(1-x) \ln^2 x \; dx = -\frac{1}{(a+1)^4}[4 + 2(a+1)\gamma + 2(a+1)\psi(a+1)$$

$$-2(a+1)^2 \psi'(a+1) + (a+1)^3 \psi''(a+2)], \qquad \text{Re } a > -1, \tag{A.2}$$

may be obtained from logarithmic differentiation twice with respect a of the integral

$$\int_0^1 x^a \ln(1-x) dx = -\frac{1}{a+1} H_{a+1} = -\frac{1}{a+1}[\psi(a+2) + \gamma], \qquad \text{Re } a > -1, \tag{A.3}$$

along with the functional equations of ψ and ψ'. We then expand the integral of the left side of (A.1) as

$$\int_0^1 \ln(1-x)\frac{\ln^2 x}{1+x}dx = \sum_{j=0}^{\infty}(-1)^j \int_0^1 x^j \ln(1-x)\ln^2 x \ dx$$

$$= -\sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)^4}[4+2(j+1)\gamma+2(j+1)\psi(j+1)$$

$$-2(j+1)^2\psi'(j+1)+(j+1)^3\psi''(j+2)]. \tag{A.4}$$

By using the summation form of I_2 in (12) and the alternating form of the Riemann zeta function, the Proposition follows.

For the polygammic summations in (A.1), we have the following integral representations:

$$\sum_{j=1}^{\infty}\frac{(-1)^j}{j^3}\psi(j+1) = \frac{1}{4}\int_0^1[4\mathrm{Li}_3(-t)+3\zeta(3)]\frac{dt}{t-1}+\frac{3}{4}\gamma\zeta(3), \tag{A.5}$$

and

$$\sum_{j=1}^{\infty}\frac{(-1)^j}{j^2}\psi'(j) = \int_0^1\frac{\mathrm{Li}_2(-t)}{t(t-1)}\ln t \ dt. \tag{A.6}$$

These follow from using a standard integral representation for ψ [8] (p. 943). They may be rewritten in a number of ways by using integration by parts.

In addition, we have

$$\sum_{j=1}^{\infty}\frac{(-1)^j}{j^3}\psi(j) = \frac{1}{2}\sum_{j=1}^{\infty}(-1)^j\psi(j)\int_0^{\infty}t^2e^{-jt}dt$$

$$= \frac{1}{2}\int_0^{\infty}\frac{t^2}{1+e^t}[\gamma+\ln(1+e^{-t})]dt = \frac{1}{2}\int_0^1\frac{\ln^2 u}{1+u}[\gamma+\ln(1+u)]du$$

$$= \frac{3}{4}\gamma\zeta(3)+\frac{1}{48}\left[-\pi^4-4\pi^2\ln^2 2+4\ln^4 2+96\mathrm{Li}_4\left(\frac{1}{2}\right)+84\ln 2\ \zeta(3)\right]. \tag{A.7}$$

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This result may also be obtained from (30) of [4]. Moreover,

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} \psi'(j) = \sum_{j=1}^{\infty} (-1)^j \psi'(j) \int_0^{\infty} t e^{-jt} dt$$

$$= -\frac{5}{4}\zeta(4) + \int_0^{\infty} \frac{t \operatorname{Li}_2(-e^{-t})}{1 + e^t} dt$$

$$= -\frac{5}{4}\zeta(4) - \int_0^1 \frac{\operatorname{Li}_2(-u) \ln u}{1 + u} du, \qquad (A.8)$$

whereby integrating by parts we have

$$\int_0^1 \frac{\operatorname{Li}_2(-u) \ln u}{1 + u} du = -\int_0^1 [\ln(1 + u) \ln u - \operatorname{Li}_2(-u)] \frac{\ln(1 + u)}{u} du$$

$$= \frac{1}{24}\left[\pi^4 + 4\pi^2 \ln^2 2 - 4\ln^4 2 - 96\operatorname{Li}_4\left(\frac{1}{2}\right) - 84\ln 2\ \zeta(3)\right] + \frac{\pi^4}{288}. \qquad (A.9)$$

By expanding the factor $\ln(1 - x)$ of the left side of (A.1) in power series we also have

$$\int_0^1 \ln(1 - x)\frac{\ln^2 x}{1 + x} dx = -\frac{1}{8}\sum_{k=1}^{\infty} \frac{1}{k}\left[\psi''\left(\frac{k}{2} + 1\right) - \psi''\left(\frac{k + 1}{2}\right)\right]. \qquad (A.10)$$

Related tetragamma series are given by

$$\sum_{k=1}^{\infty} \frac{1}{k}\psi''\left(\frac{k}{2} + 1\right) = \sum_{k=1}^{\infty} \frac{1}{k}\psi''\left(\frac{k}{2}\right) + 16\zeta(4), \qquad (A.11)$$

and

$$\frac{1}{8}\sum_{k=1}^{\infty} \frac{1}{k}\left[\psi''\left(\frac{k}{2}\right) + \psi''\left(\frac{k + 1}{2}\right)\right] = \sum_{k=1}^{\infty} \frac{\psi''(k)}{k} = -\frac{5}{2}\zeta(4). \qquad (A.12)$$

For the latter relation, we have applied the duplication formula

$$\psi''(2x) = \frac{1}{8}\left[\psi''(x) + \psi''\left(x + \frac{1}{2}\right)\right]. \qquad (A.13)$$

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We may use (42) to show the equivalence of the right side of (A.7) to Proposition A1. For by applying the operator ∂_b^2 to both sides of (42) we have

$$2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+b)^3} = \frac{1}{8}\left[\psi''\left(\frac{b+1}{2}\right) - \psi''\left(\frac{b}{2}+1\right)\right]. \tag{A.14}$$

We then have with the aid of partial fractions

$$\frac{1}{8}\sum_{b=1}^{\infty}\frac{1}{b}\left[\psi''\left(\frac{b+1}{2}\right) - \psi''\left(\frac{b}{2}+1\right)\right] = 2\sum_{b=1}^{\infty}\frac{1}{b}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n+b)^3}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}[2\gamma + 2\psi(n+1) - 2n\psi'(n+1) + n^2\psi''(n+1)]$$

$$= -\frac{3}{2}\gamma\zeta(3) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}[\psi(n+1) - n\psi'(n+1)] + I_2$$

$$= -\frac{3}{2}\gamma\zeta(3) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}[\psi(n) - n\psi'(n)] + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} + I_2$$

$$= -\frac{7\pi^4}{180} - \frac{3}{2}\gamma\zeta(3) + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}[\psi(n) - n\psi'(n)] + I_2. \tag{A.15}$$

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